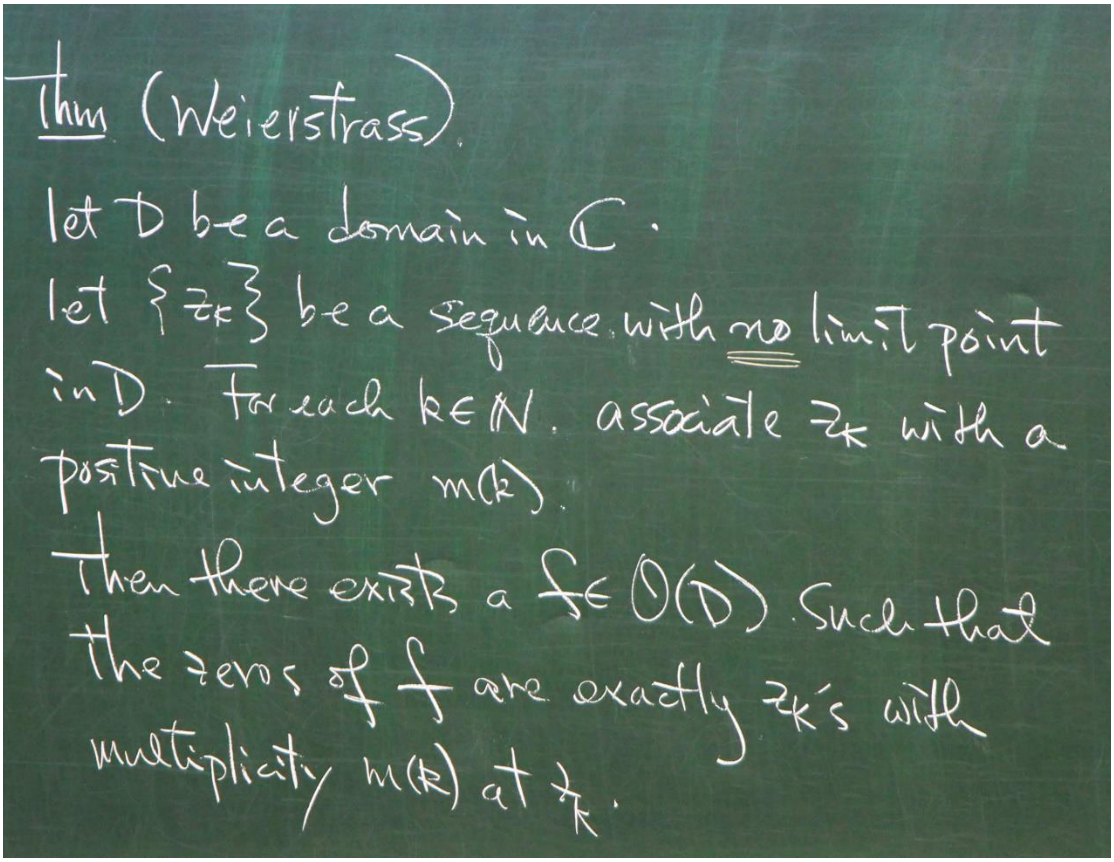
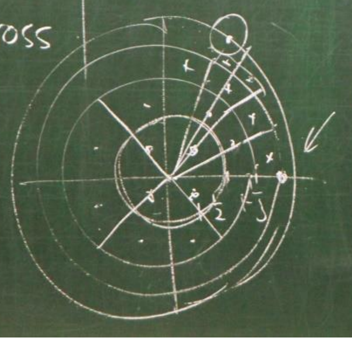
【10920 程守慶教授複變數函數論 / 第 4 堂版書】



Thm. Let D be a domain in $\mathbb{C}$.
Then there exists a $f \in \mathcal{O}(D)$.
such that f can <u>not</u> be
holomorphically extended across
any boundary point.

ain in $\mathbb{C}$.
$f \in \mathcal{O}(D)$.
ot be
nded across

pf. Choose a seq. of points $\{z_k\}$ with no limit in D.
such that every boundary point is a limit point
of this sequence.

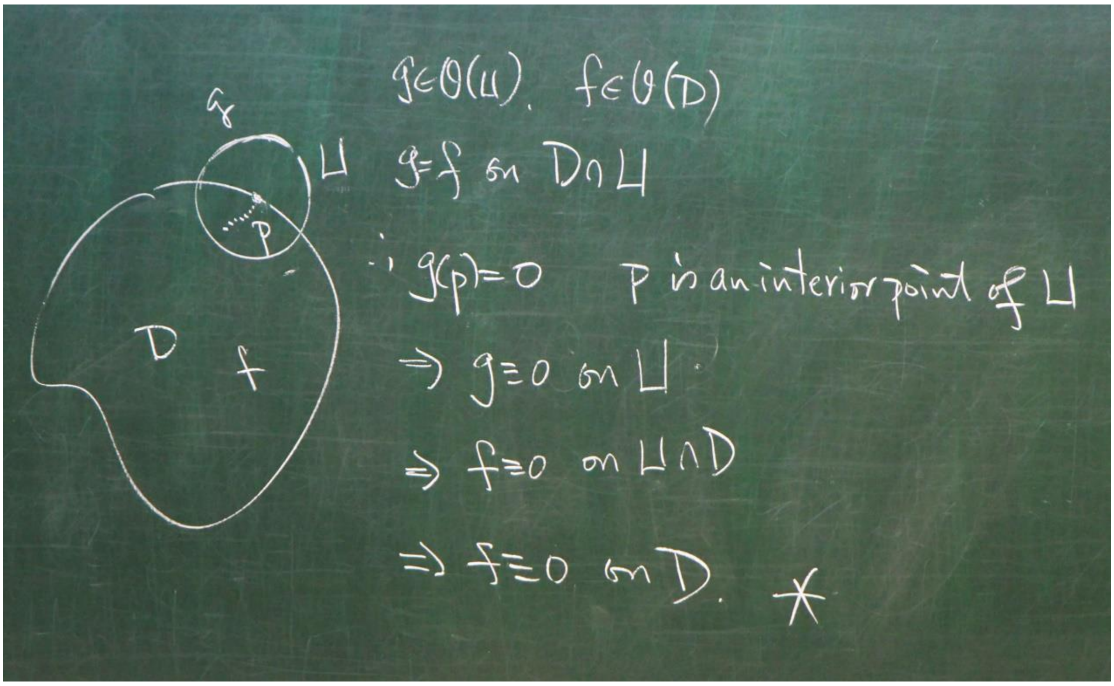


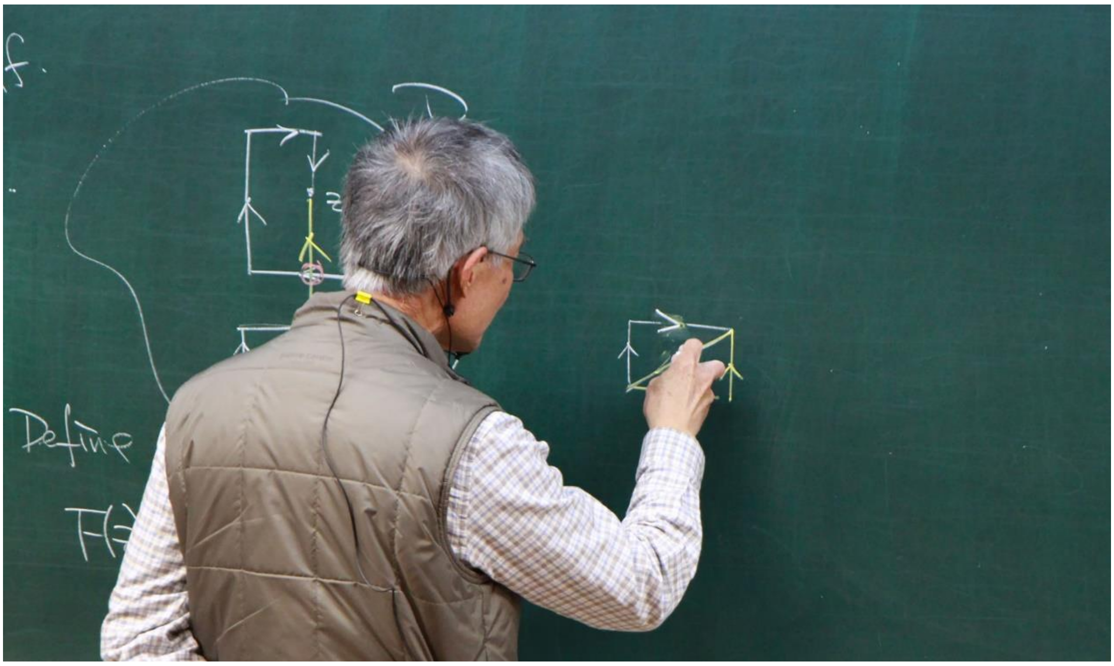
$1 - \frac{1}{k}$

$\therefore$ By Weierstrass' thm.

$\exists f \in \mathcal{O}(D)$ s.t.

$Z_f = \{z_k\}_{k=1}^{\infty}$ $\quad$ $m(k) = 1$


g ∈ O(U). f ∈ O(D)
g = f on D ∩ U
∵ g(p) = 0   p is an interior point of U
⇒ g ≡ 0 on U.
⇒ f ≡ 0 on U ∩ D
⇒ f ≡ 0 on D.
g
U
p
D
f


Define

Thm. (Morera).

$D \subseteq \mathbb{C}$ simply-connected domain.

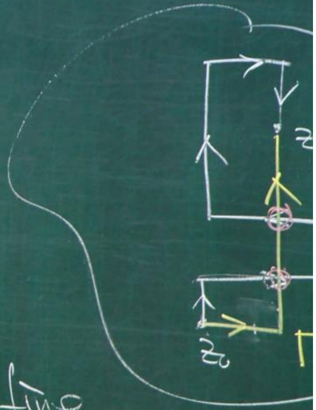
let $f \in C(D)$.

If $\oint_{\partial\Delta} f(z)\,dz = 0$,

for any triangle Δ in D.

Then $f \in \mathcal{O}(D)$

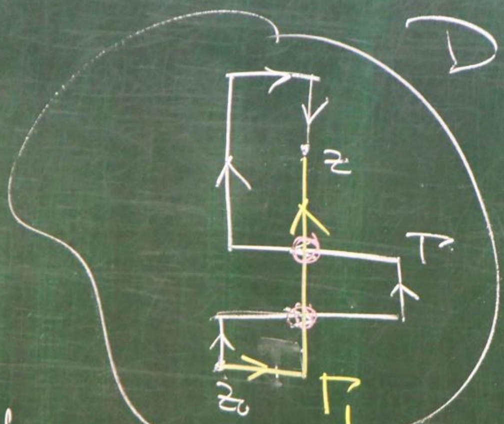
pf.



Define

$$F(z) = \int_{\Gamma} f(z)\,dz$$

pf.



main.

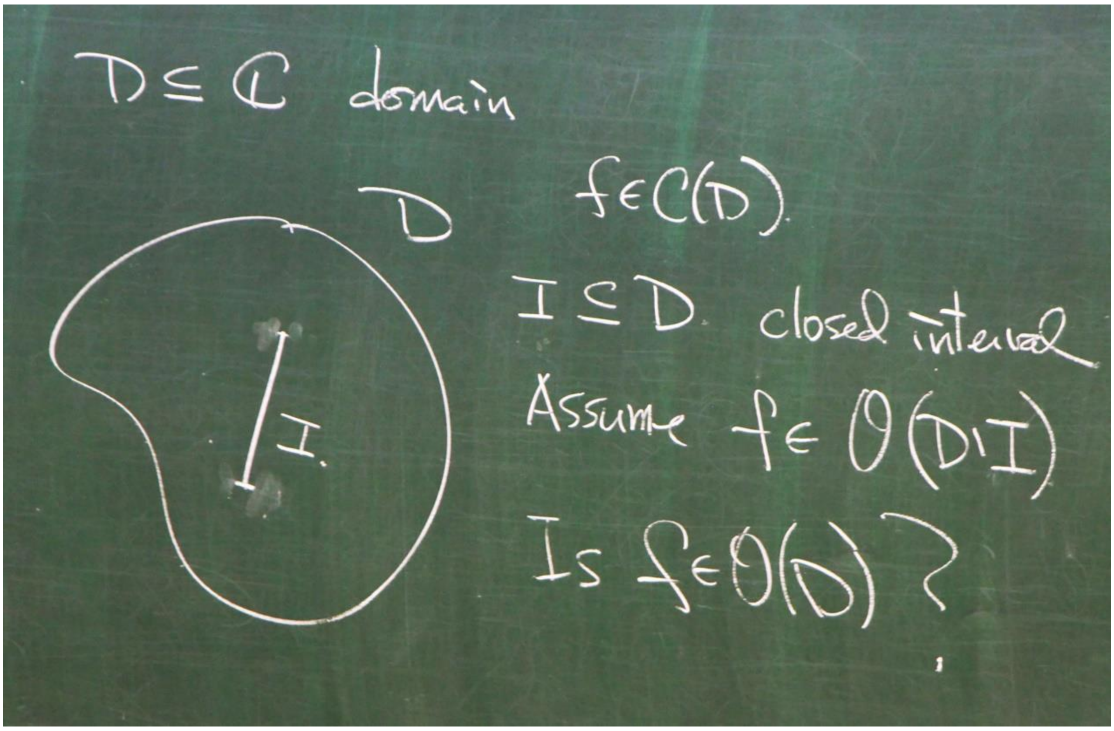
Define

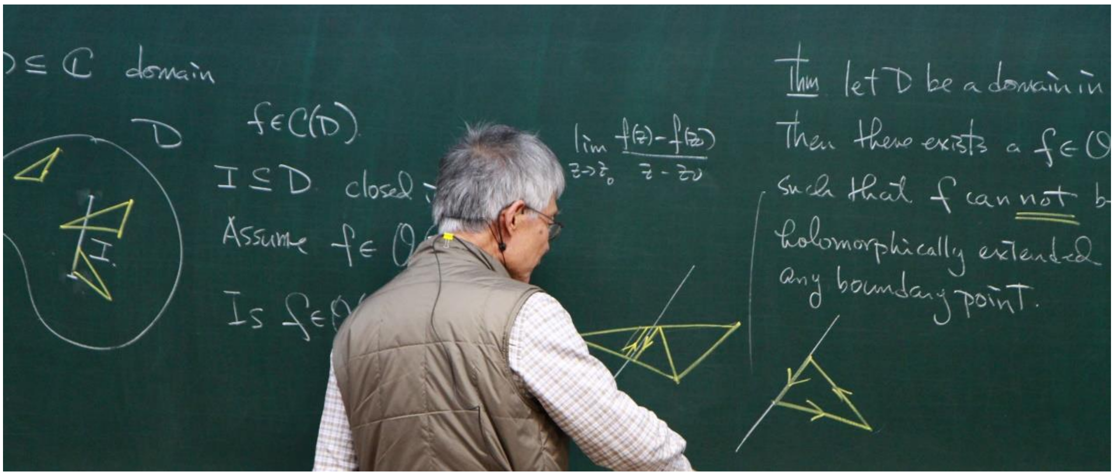
$$F(z) = \int_{\Gamma} f(z)\,dz$$

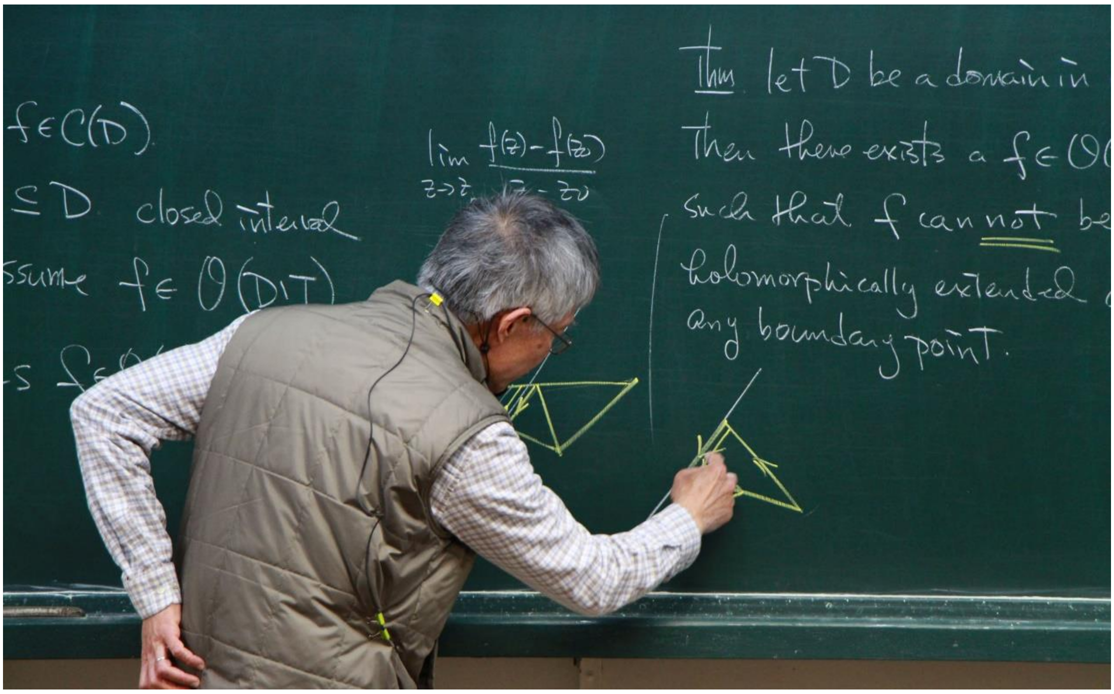
$\therefore F'(z) = f(z)$

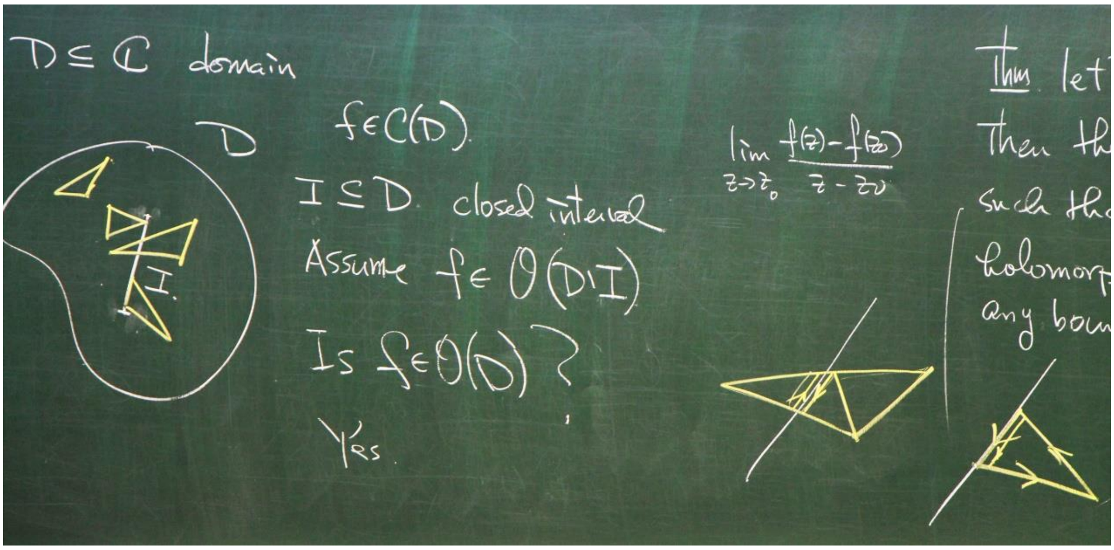
$\therefore F \in \mathcal{O}(D)$

$\therefore f \in \mathcal{O}(D)$


D ⊆ ℂ domain
D
f ∈ C(D).
I ⊆ D. closed interval
I.
Assume f ∈ O(D\I)
Is f ∈ O(D) ?


D ⊆ ℂ domain
D
f ∈ C(D).
I ⊆ D. closed
Assume f ∈
Is f ∈
lim f(z) - f(z0) / z - z0
z→z0
Thm let D be a domain in
Then there exists a f ∈
such that f can not b
holomorphically extended
any boundary point.


Thm let D be a domain in
Then there exists a $f \in \mathcal{O}($
such that f can not be
holomorphically extended
any boundary point.
$f \in C(D)$
$\subseteq D$ closed interval
ssume $f \in \mathcal{O}(D \setminus I)$
$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$


$D \subseteq \mathbb{C}$ domain
D
I
$f \in C(D)$
$I \subseteq D$ closed interval
Assume $f \in \mathcal{O}(D \setminus I)$
Is $f \in \mathcal{O}(D)$?
Yes
$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$
Thm let
Then th
such tha
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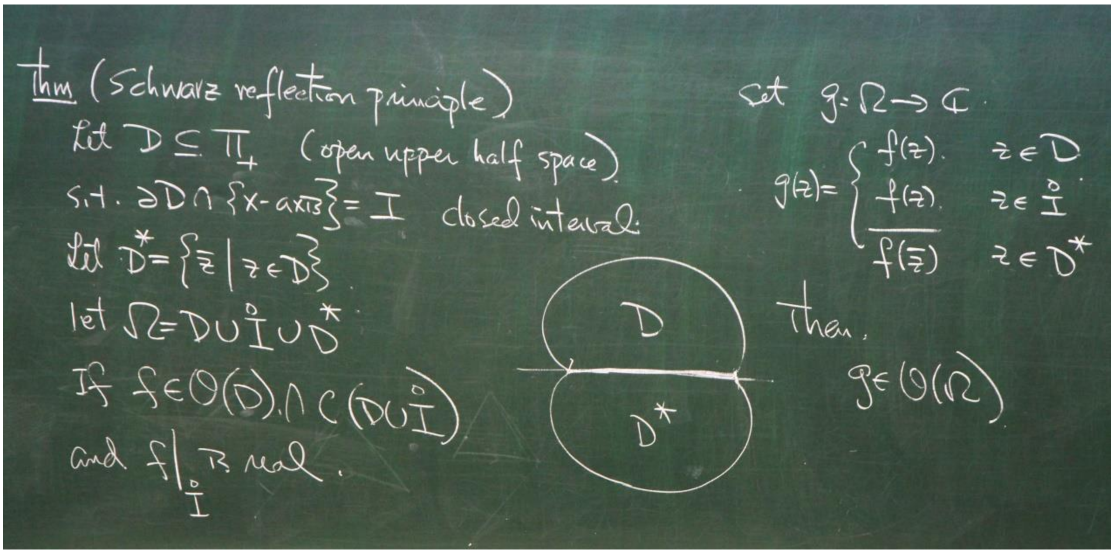
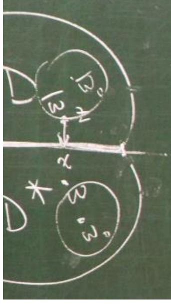

thm (Schwarz reflection principle)
let $D \subseteq \Pi_+$ (open upper half space)
s.t. $\partial D \cap \{x\text{-axis}\} = I$ closed interval
let $D^* = \{\bar{z} \mid z \in D\}$
let $\Omega = D \cup \mathring{I} \cup D^*$
If $f \in \mathcal{O}(D) \cap C(D \cup \mathring{I})$
and $f|_{\mathring{I}}$ is real.
set $g: \Omega \to \mathbb{C}$
$g(z) = \begin{cases} f(z) & z \in D \\ f(z) & z \in \mathring{I} \\ \overline{f(\bar{z})} & z \in D^* \end{cases}$
Then
$g \in \mathcal{O}(\Omega)$
D
D^*

Set $g:\Omega \to \mathbb{C}$.

$$g(z)=\begin{cases} f(z) & z\in D \\ f(z) & z\in \mathring{I} \\ \overline{f(\bar z)} & z\in D^* \end{cases}$$



Then, $g\in \mathcal{O}(\Omega)$

$$\underline{f(z)=u(z)+iv(z)} \qquad z\to x$$

$$f(x)=u(x)$$

$$f(w)=u(\bar w)-iv(\bar w) \qquad w\to x$$

$$\bar w \to X$$

$$f(x)=u(x)$$

$$\bar w_0$$

$$\boxed{\bar w\in D}$$

$$f(z)=\sum a_k (z-\bar w_0)^k$$

$$z\to x$$

$$w\in D^*$$

$$g(w) = \overline{f(\bar w)} = \overline{\sum a_k(\bar w-\bar w_0)^k} = \sum \bar a_k (w-w_0)^k$$

$$w\to x$$

$$\bar w\to X$$

(Schwarz reflection principle)

$D \subseteq \Pi_+$ (open upper half space).

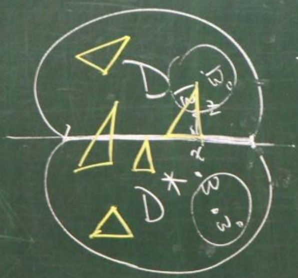
$\partial D \cap \{x\text{-axis}\} = I$ closed interval.

$D^* = \{\bar z \mid z \in D\}$

$\Omega = D \cup \mathring{I} \cup D^*$

$f \in \mathcal{O}(D) \cap C(D \cup \mathring{I})$

$f|_{\mathring{I}}$ is real.



Set $g: \Omega \to \mathbb{C}$.

$$g(z) = \begin{cases} f(z). & z \in D \\ f(z). & z \in \mathring{I} \\ \overline{f(\bar z)} & z \in D^* \end{cases}$$

then. $g \in \mathcal{O}(\Omega)$

$z \to x$

$w \to x$

$\bar w \to x$

$$w \in D^*$$

$$g(w) = \overline{f(\bar w)} = \overline{\sum a_k (\bar w - \bar w_0)^k} = \sum \bar a_k (w - w_0)^k$$

$\bar w_0$

$\boxed{\bar w \in D}$

$$f(z) = \sum a_k (z - \bar w_0)^k$$

Singular point

奇異點

$D \subseteq \mathbb{C}$ domain.

$z_0 \in D$

If $f \in \mathcal{O}(D \setminus \{z_0\})$

z_0 : isolated singular point.

singularity.

(I) $\exists M > 0$. $|f(z)| \leq M$ $\forall z \in D \setminus \{z_0\}$

(removable singularity).

(II) $\lim\limits_{z \to z_0} |f(z)| = +\infty$ (pole) 極点.

(III) f is not bounded on $D \setminus \{z_0\}$

and $\lim\limits_{z \to z_0} |f(z)| \neq +\infty$ (essential singularity (本質奇異点))

Def. Let D be a domain in $\mathbb{C}$, $E \subseteq D$. We say E is a set of uniqueness if, for $f \in \mathcal{O}(D)$, $f(z)=0$, $z \in E$ implies $f(z) \equiv 0$ $\forall z \in D$.

Def. Let D be a domain in $\mathbb{C}$, $E \subseteq D$. We say E is a set of uniqueness if, for $f \in \mathcal{O}(D)$, $f(z)=0$, $z \in E$ implies $f(z) \equiv 0$ $\forall z \in D$.

$\sin \pi z$

Any point sequence with an accumulation point in D is a set of uniqueness.

Any open subset of D

is a set of uniqueness.

<u>Thm</u> (identity theorem) 恒等定理.

Let $f, g \in \mathcal{O}(D)$. $D \subseteq \mathbb{C}$. domain

If $f = g$ on an open subset of D.

Then $f \equiv g$ on D